

# Birational geometry of defective varieties

Edoardo Ballico and Claudio Fontanari

## Abstract

Here we investigate the birational geometry of projective varieties of arbitrary dimension having defective higher secant varieties. We apply the classical tool of tangential projections and we determine natural conditions for uniruledness, rational connectivity, and rationality.

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## 1 Introduction

We work over an algebraically closed field  $\mathbb{K}$  with  $\text{char}(\mathbb{K}) = 0$ . Let  $X \subset \mathbb{P}^r$  be an integral nondegenerate  $n$ -dimensional variety. Recall that for every integer  $k \geq 0$  the  $k$ -secant variety  $S^k(X)$  is defined as the Zariski closure of the set of the points in  $\mathbb{P}^r$  lying in the span of  $k + 1$  independent points of  $X$ . An easy parameter count shows that the expected dimension of  $S^k(X)$  is exactly  $\min\{r, n(k + 1) + k\}$ . However, there are natural examples of projective varieties having secant varieties of strictly lower dimension: for instance, the first secant variety of the 2-Veronese embedding of  $\mathbb{P}^2$  in  $\mathbb{P}^5$  has dimension 4 instead of 5. More generally, one defines the  $k$ -defect  $\delta_k(X) = \min\{r, n(k + 1) + k\} - \dim S^k(X)$  and says that  $X$  is  $k$ -defective if  $\delta_k(X) \geq 1$ . It seems reasonable to regard defective varieties as exceptional and try to classify them. The first result in this direction, stated by Del Pezzo in 1887 and proved by Severi in 1901 (see [5], [10], and also [4] for a modern proof), characterizes the 2-Veronese of  $\mathbb{P}^2$  as the unique 1-defective surface which is not a cone. Along the same lines, subsequent contributions by Palatini ([6] and [7]), Scorza ([8] and [9]), and Terracini ([11] and [12]) set up the classification of  $k$ -defective surfaces and of 1-defective varieties in dimension up to four. This classical work has been recently reconsidered

and generalized by Chiantini and Ciliberto in [1] and [2]. It turns out that one of the main tools for understanding defective varieties is provided by the technique of tangential projections. Namely, assume that  $X$  is not  $(k-1)$ -defective and let  $p_1, \dots, p_k$  be general points on  $X \subset \mathbb{P}^r$ . The general  $k$ -tangential projection  $\tau_{X,k}$  is the projection of  $X$  from the linear span of its tangent spaces at  $p_1, \dots, p_k$ . By the classical Lemma of Terracini (see [11] and [3] for a modern version),  $X_k := \tau_{X,k}(X)$  is lower dimensional than  $X$  if and only if  $X$  is  $k$ -defective. Therefore, the classification of defective varieties reduces to the classification of varieties which drop dimension in the general tangential projection. However, we believe that the only reasonable goal in arbitrary dimension is to determine some geometrical properties of defective varieties, and the present paper is indeed a first attempt in this direction. In order to state our main results, we recall from [1] that the *contact locus* of a general hyperplane section  $H$  tangent at  $k+1$  general points  $p_1, \dots, p_{k+1}$  of  $X$  is the union  $\Sigma$  of the irreducible components of the singular locus of  $H$  containing  $p_1, \dots, p_{k+1}$ . One defines  $\nu_k(X) := \dim \Sigma$  and says that  $X$  is  *$k$ -weakly defective* whenever  $\nu_k(X) \geq 1$ . The reason for this terminology is simply that a  $k$ -defective variety is always weakly defective (indeed, we are going to show in Proposition 1 that  $\nu_k(X) \geq \delta_k(X)$ ), but the converse is not true (look at cones). We point out the following:

**Lemma 1.** *Fix integers  $k \geq 1$ ,  $n \geq 2$ ,  $r \geq (k+1)(n+1)$ , and let  $X \subset \mathbb{P}^r$  be an integral nondegenerate  $n$ -dimensional variety. If  $X$  is not  $k$ -weakly defective, then  $\tau_{X,k}$  is birational.*

Moreover, for every  $k \geq 1$  and  $n \geq 2$  we exhibit a projective  $n$ -dimensional variety being  $k$ -weakly defective but not  $k$ -defective such that  $\tau_{X,k}$  is not birational (see Example 1). Next, as an application of Lemma 1, we investigate the birational geometry of a  $k$ -defective variety of arbitrary dimension:

**Theorem 1.** *Fix integers  $k \geq 1$ ,  $n \geq 2$ ,  $r \geq (k+1)(n+1)-1$ , and let  $X \subset \mathbb{P}^r$  be an integral nondegenerate  $n$ -dimensional variety which is  $k$ -defective but not  $(k-1)$ -defective. Assume that  $\nu_k(X) = \delta_k(X)$  and, for  $k \geq 2$ , that  $X$  is not  $(k-1)$ -weakly defective. Then  $X$  is uniruled. Assume moreover that the general contact locus of  $X$  is irreducible. Then  $X$  is rationally connected and for  $\nu_k(X) = \delta_k(X) = 1$  it is rational.*

We stress that the hypotheses for uniruledness cannot be removed: in Examples 2 and 3 we collect a series of non-uniruled defective varieties of any dimension which do not satisfy exactly one of the listed assumptions. An analogous remark applies to the second part of the statement, where irreducibility turns out to be essential. For instance, if  $X$  is a cone over a

curve of positive genus, then two general points on  $X$  cannot be joined by a rational curve. Indeed, we suspect that a defective variety with  $\nu_k = \delta_k$  and reducible general contact locus should be a cone (this is certainly the case in dimension up to four by [2], proof of Proposition 4.2, and [9], § 14).

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## 2 The proofs

Let  $k$  be the minimal integer such that  $X$  is  $k$ -defective. Then  $\tau_{X,k-1}(X)$  is generically finite and by definition we have

$$\nu_k(X) = \nu_1(X_{k-1}). \quad (1)$$

Moreover, by applying twice the equality  $\delta_h(Y) = \dim Y - \dim \tau_{Y,h}(Y)$ , first with  $Y = X$  and  $h = k$ , then with  $Y = X_{k-1}$  and  $h = 1$ , we deduce

$$\delta_k(X) = \delta_1(X_{k-1}). \quad (2)$$

**Proposition 1.** *Fix integers  $k \geq 1$ ,  $n \geq 2$ ,  $r \geq (k+1)(n+1) - 1$ , and let  $X \subset \mathbb{P}^r$  be an integral nondegenerate  $n$ -dimensional variety. Assume that  $k$  is the minimal integer such that  $X$  is  $k$ -defective. Then we have  $\nu_k(X) \geq \delta_k(X)$ .*

*Proof.* Assume first  $k = 1$ . If  $\delta_1(X) = 1$ , we are just claiming that  $\nu_1(X) > 0$ , which is well-known (see for instance [1], Theorem 1.1). If instead  $\delta_1(X) \geq 2$ , we take a general hyperplane section  $H$ . By [2], Lemma 3.6, we have  $\nu_1(H) = \nu_1(X) - 1$  and  $\delta_1(H) = \delta_1(X) - 1$ , so we conclude by induction. Assume now  $k > 1$ . By (1) and (2), we are reduced to the previous case, so the proof is over. □

*Proof of Lemma 1.* Recall that  $\tau_{X,k}$  is defined as the projection of  $X$  from  $\langle T_{p_1}(X), \dots, T_{p_k}(X) \rangle$ , where  $p_i$  is a general point on  $X$  and  $T_{p_i}(X)$  denotes the tangent space to  $X$  at  $p_i$ . Pick a general point  $p_{k+1} \in X$  and let  $q = \tau_{X,k}(p_{k+1})$ . By Proposition 1,  $X$  is not  $k$ -defective, so  $\tau_{X,k}$  is generically finite. Therefore  $\tau_{X,k}^{-1}(q) = \{p_{k+1}, \dots, p_{k+d}\}$  and we want to show that  $d = 1$ . Since  $\tau_{X,k}(T_{p_{k+h}}(X)) = T_q(X_k)$  for every  $h$  with  $1 \leq h \leq d$ , it follows that  $\langle T_{p_1}(X), \dots, T_{p_k}(X), T_{p_{k+h}}(X) \rangle$  does not depend on  $h$ . On the other hand, by [1], Theorem 1.4, the general hyperplane which is tangent to  $X$  at  $p_1, \dots, p_{k+1}$  is not tangent to  $X$  at any other point, so we have  $d = h = 1$  and the proof is over. □

In order to construct nontrivial examples of projective varieties whose general  $k$ -tangential projection is not birational, we are going to apply the following criterion:

**Lemma 2.** *Fix integers  $k \geq 1$ ,  $n \geq 2$ ,  $r \geq (k+1)(n+1) - 1$ , and let  $X \subset \mathbb{P}^r$  be an integral nondegenerate  $n$ -dimensional variety. Assume that  $X$  is  $k$ -weakly defective, but not  $k$ -defective. If  $X$  is not uniruled, then  $\tau_{X,k}(X)$  is generically finite but not birational.*

*Proof.* Notice that  $X$  is a fortiori not  $(k-1)$ -defective, so we can apply Proposition 3.6 in [1] with  $h = k$  to obtain that  $X_k$  is 0-defective. Hence  $X_k$  is a developable scroll (see for instance [1], Remark 3.1 (ii)), in particular it is uniruled and it follows that  $X_k$  is not birational to  $X$ . □

**Example 1.** Let  $k \geq 1$ ,  $n \geq 2$ , and  $r \geq (k+1)(n+1) - 1$ . Take a  $(n-1)$ -dimensional variety  $C$  in  $\mathbb{P}^r$  which is not uniruled, a linear space  $V \subset \mathbb{P}^r$  with  $\dim V = k$ , and a smooth hypersurface  $H \subset \mathbb{P}^r$  of degree  $d \geq k+2$  such that  $V \not\subset H$ . Let  $W$  be the cone over  $C$  with vertex  $V$  and define  $X := H \cap W$ . By [1], Example 4.3,  $X$  is  $k$ -weakly defective but not  $k$ -defective. We claim that  $X$  is not uniruled. Let  $\pi : X \rightarrow C$  be the projection and take a general point  $p \in X$ . If  $R \subseteq X$  is a rational curve passing through  $p$ , then  $R$  is contained in a fiber of  $\pi$ , otherwise its projection would be a rational curve through a general point of  $C$ , in contradiction with the non-uniruledness of  $C$ . On the other hand, the general fiber of  $\pi$  is set-theoretically the intersection of  $H$  with a  $(k+1)$ -dimensional linear space, hence it is a smooth hypersurface of high degree, which is not covered by rational curves. Hence the claim is established and from Lemma 2 we deduce that  $\tau_{X,k}(X)$  is generically finite but not birational.

*Proof of Theorem 1.* Assume first  $k = 1$ . If  $\nu_1(X) = \delta_1(X) = 1$ , from the so-called Terracini's Theorem ([1], Theorem 1.1) it follows that the general contact locus  $\Sigma$  of  $X$  imposes only three conditions on hyperplanes containing it, in particular it is a plane curve. Moreover, if  $\Sigma$  had degree  $d > 2$ , then the general secant line to  $X$  would be a multisequant, which is a contradiction (the above argument is borrowed from [2], proof of Proposition 4.2). Hence  $X$  is uniruled and if  $\Sigma$  is irreducible then  $X$  is rationally connected. More precisely, if  $d = 1$  then two general points on  $X$  would be joined by a straight line, a contradiction since  $X$  is nondegenerate. Therefore if we fix a general  $p \in X$  and for general  $q \in X$  we assume that the corresponding contact locus  $\Sigma_{pq}$  is irreducible, then  $\Sigma_{pq}$  is a smooth conic and a natural birational map from  $X$  to its tangent space  $T_p(X) \cong \mathbb{P}^n$  is defined by sending  $q$  to the intersection point between the tangent lines to  $\Sigma_{pq}$  at  $p$  and  $q$  (see [9],

§ 15). If instead  $\delta_1(X) \geq 2$ , let  $H$  be a general hyperplane section. By [2], Lemma 3.6, we have  $\nu_1(H) = \nu_1(X) - 1$  and  $\delta_1(H) = \delta_1(X) - 1$ , so uniruledness and rational connectivity follow by induction. Assume now  $k > 1$ . By (1) and (2), the previous cases apply to  $X_{k-1}$ . On the other hand, from Lemma 1 we get a birational map between  $X$  and  $X_{k-1}$ , so the proof is over. □

**Example 2.** Here we show that the assumption  $X$  not  $(k-1)$ -weakly defective is essential for every  $n \geq 2$ . Fix  $r \geq 3n+2$  and let  $C \subset \mathbb{P}^r$  be a  $(n-1)$ -dimensional variety which is not uniruled. Take a line  $L \subset \mathbb{P}^r$  and a smooth hypersurface  $H \subset \mathbb{P}^r$  of degree  $d \geq 3$  such that  $L \not\subset H$ . Let  $W$  be the cone over  $C$  with vertex  $V$  and define  $X := H \cap W$ . By [1], Example 4.3, we have  $\delta_2(X) = \nu_2(X) = 1$  and  $X$  is also 1-weakly defective. Moreover, arguing as in Example 1, it is easy to check that  $X$  is not uniruled.

Finally we focus on the assumption  $\nu_k(X) = \delta_k(X)$ . By Proposition 1, it is always satisfied in the case of surfaces; however, in higher dimension it is no more automatic.

**Example 3.** As in [2], Example 2.2, we consider a  $n$ -dimensional variety  $X$  contained in a  $(n+1)$ -dimensional cone  $W$  over a curve  $C$  in  $\mathbb{P}^r$  with  $r \geq 2n+1$ . We have  $\delta_1(X) = n-2$  and  $\nu_1(X) = n-1$ ; moreover, if we assume  $g(C) \geq 1$  and we take  $X := H \cap W$ , where  $H$  is a general hypersurface in  $\mathbb{P}^r$  of high degree, then the same argument as in Example 1 shows that  $X$  is not uniruled.

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Edoardo Ballico  
 Università degli Studi di Trento  
 Dipartimento di Matematica  
 Via Sommarive 14  
 38050 Povo, Trento, Italy  
 e-mail: ballico@science.unitn.it

Claudio Fontanari  
 Università degli Studi di Trento  
 Dipartimento di Matematica  
 Via Sommarive 14  
 38050 Povo, Trento, Italy  
 e-mail: fontanar@science.unitn.it